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LOCALIZATION ANALYSIS VIA A GEOMETRICAL METHOD

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Abstract—A geometrical technique is proposed in order to solve explicitly the critical conditions at localization for a quite general constitutive behaviour with isotropic elastic properties. These critical conditions are shown to be closely related to the spectral properties (eigenvalues and eigenvectors) of the sum and difference of two tensors describing the inelastic effects. When these two tensors are coaxial, it is shown that the normal to a potential localization plane always lies in one of their principal planes. It is also demonstrated that, depending on the constitutive behaviour and the loading conditions, several expressions for the critical hardening modulus at localization are available and their respective domain of validity well defined. The roles and interactions of both deviatoric and hydrostatic non-associativities in the critical conditions of localization are emphasized.

1. INTRODUCTION

A large amount of research effort has been and is still being devoted to characterize the transition from a continuous deformation field to a discontinuous one in inelastic bodies and to follow the real behaviour up to failure. In the pioneering works of Hill (1962), Rice (1976), Rudnicki and Rice (1975), and Rice and Rudnicki (1980), the abrupt changes in the deformation field were understood as an instability of the material and described as a bifurcation in modes involving jumps of the velocity gradient across a planar surface; in the same works necessary conditions for this localization phenomenon to occur were derived. Borré and Maier (1989) proved these conditions were sufficient. The localization condition was also shown to correspond to the loss of ellipticity of the governing field equations.

The purpose of the present paper is to obtain closed form solutions for localization conditions for a very general class of rate independent material models, including non-associative elastoplastic and isotropic damage models.

Interest in determining analytically the onset of localization lies mainly in the following considerations.

(1) For a given material model the explicit expression of the critical hardening modulus as a function of the stress state allows simple loading paths to be selected (for instance, radial loading paths) which are less favourable to localization, and can therefore be chosen in experiments designed to identify material parameters concerning the homogeneous response.

(2) Explicit solutions of the localization condition in terms of critical hardening modulus, critical directions and form of the strain rate jump highlight the influence of the main features of a material model (volumetric associativity, deviatoric associativity, state couplings, etc.) at the onset of localization. If experimental data on the localized response of the material are available, this is very useful in order to build a realistic model [see e.g. Comi *et al.* (1994)].

(3) In the context of finite element analysis conceived to capture localized deformation patterns, the onset of localization must be determined at each Gauss point in order to realign the mesh or to enrich the elements with special functions [see e.g. Ortiz *et al.* (1987)]. Since the localization condition must be checked at each Gauss point, for each load step, the use of a numerical algorithm of optimization to solve the localization condition can be very time consuming. Hence, analytical solutions greatly simplify the analysis.

Analytical solutions have been obtained in various contexts and by various authors. Thus, in analysing bifurcation phenomena in the plane strain test, Hill and Hutchinson (1975) defined the different regimes of the field equations and gave the conditions for the loss of ellipticity for the most general constitutive model under the constraint of incompressibility when the increment of shear strain is independent of the increment of hydrostatic stress. Their results were generalized later by Needleman (1979) for a particular choice in this dependence allowing for non-normality. Again, in plane strain conditions, analysing shear band formation, Vardoulakis (1980) furnished analytical solutions for models describing sand behaviour, while Hutchinson and Tvergaard (1981) provided closed form solutions for three material models, namely non-linear elasticity, kinematic hardening plasticity and a plasticity theory based on yield surface corner development. In dealing with localized necking in thin sheets, Storen and Rice (1975) also furnished the conditions for the loss of ellipticity in plane stress for a deformation theory of plasticity used as a model for flow theory at a yield vertex: these conditions were given in closed form only for some particular situations.

For three-dimensional problems, using the Lagrange multipliers method, the localization conditions were solved by several authors, but for models involving only volumetric non-associativity or under simple loading conditions [see e.g. Rudnicki and Rice (1975); Runesson *et al.* (1991); Ottosen and Runesson (1991a)]. Ottosen and Runesson (1991b) considered a particular model, namely a Mohr–Coulomb model involving deviatoric non-normality. However, Bigoni and Hueckel (1991) derived closed form solutions, by the Lagrange multiplier method, for the same general class of material models (containing deviatoric and volumetric non-associativities) that will be considered hereafter.

In this paper a completely different method based on a geometric interpretation of the localization condition is used; this method leads to finer *a priori* results. The geometric method, in a slightly different form, was first proposed by Benallal (1992) to analyse the roles of thermal effects on localization, and developed by Benallal and Comi (1993) to explore the role of deviatoric non-associativity. A similar technique has also been applied very recently, by Perrin and Leblond (1994), to the Drucker–Prager model.

In Section 2 the constitutive behaviour considered and the corresponding localization conditions are formulated. In Section 3, the geometric interpretation of the localization condition is presented. Under the assumption of coaxiality of the gradients of the loading surface and of the inelastic potential, closed form solutions of localization conditions are found and the domain of validity of each expression is given explicitly. The possible continuous localization (i.e. the material is in loading condition on the two sides of the discontinuity surface) and/or discontinuous localization (i.e. the material is in loading condition on one side and is in unloading condition on the other side of the discontinuity surface) at the beginning of the inelastic process are analysed on the basis of the same geometric method. Section 4 is devoted to the application of the previous results to different material models belonging to the class specified in Section 2, namely a simple ductile damage model (Lemaitre, 1992) and a class of non-associative plasticity models for concrete-like materials, including the Hsieh–Tin–Chen four-parameter model and the William–Wranke five-parameter model [for a review of these models see e.g. Chen and Han (1987)].

2. CONSTITUTIVE EQUATIONS AND LOCALIZATION CONDITION

For simplicity, it will be assumed that deformations are small, i.e. geometrical effects are neglected. Attention is focused hereafter on rate independent materials, the behaviour of which can be described by the following equation:

$$\dot{\sigma} = \mathbb{L} : \dot{\epsilon}, \tag{1}$$

where σ is the stress tensor, ϵ the strain tensor and $\mathbb{L}$ the tangent modulus defined by:

$$\mathbb{L} = \begin{cases} \mathbb{E} \text{ if } f < 0 \text{ or } f = 0 \text{ and } \dot{f} < 0 \\ \mathbb{H} = \mathbb{E} - \dfrac{\alpha \otimes \beta}{G} \text{ if } f = 0 \text{ and } \dot{f} = 0. \end{cases} \tag{2}$$

In eqn (2), f is the yield function, α and β are second-order tensors, G is a scalar parameter and $\mathbb{E}$ is the current elastic tensor, assumed to be isotropic throughout the paper. A very classical form for $\mathbb{H}$ used extensively in the literature to describe non-associative elastic–plastic materials is:

$$\mathbb{H} = \mathbb{E} - \frac{\mathbb{E} : \mathbf{u} \otimes \mathbf{v} : \mathbb{E}}{g - g_s}, \tag{3}$$

where g is the so-called hardening modulus, $g_s = -\mathbf{v} : \mathbb{E} : \mathbf{u}$, the snap-back threshold modulus, and $\mathbf{u}$ and $\mathbf{v}$ are the gradients of the plastic potential and of the yield function, respectively. The hardening modulus is here assumed to be piecewise continuous.

We will denote by $\mathbf{a}$ and $\mathbf{b}$ the unit deviatoric parts of α and β and by p and q the normalized hydrostatic parts, so that:

$$\alpha = \bar{\alpha}(\mathbf{a} + p\mathbb{1}), p = \frac{\operatorname{tr}(\alpha)}{3\bar{\alpha}} \quad \beta = \bar{\beta}(\mathbf{b} + q\mathbb{1}), \quad q = \frac{\operatorname{tr}(\beta)}{3\bar{\beta}}. \tag{4}$$

$\bar{\alpha}$ and $\bar{\beta}$ are then the norms of the deviatoric parts of α and β. With these notations, the tangent modulus [eqn (2)] becomes

$$\mathbb{H} = \mathbb{E} - \frac{(\mathbf{a} + p\mathbb{1}) \otimes (\mathbf{b} + q\mathbb{1})}{H}, \tag{5}$$

with $H = G/\bar{\alpha}\bar{\beta}$. We also set $h = g/\bar{\alpha}\bar{\beta}$.

Let us then consider a homogeneous infinite body subjected to quasi-static loadings in the small strain range. The necessary and sufficient condition under which continuous localization of deformations into a planar band with normal **n** (i.e. the material is in loading inside and outside the band) can occur can be found in Rice (1976) and reads:

$$\det(\mathbf{n} \,.\, \mathbb{H} \,.\, \mathbf{n}) = 0. \tag{6}$$

A necessary condition for discontinuous localization (i.e. the material is in loading inside the band and in elastic unloading outside it) was given by Rice and Rudnicki (1980) and reads:

$$\det(\mathbf{n} \,.\, \mathbb{H} \,.\, \mathbf{n}) < 0. \tag{7}$$

Finally, Borré and Maier (1989) have demonstrated that the above conditions are actually necessary and sufficient so that the overall localization condition is:

$$\det(\mathbf{n} \,.\, \mathbb{H} \,.\, \mathbf{n}) \leqslant 0. \tag{8}$$

In general, at the beginning of a loading process the acoustic tensor is positive definite, hence the critical conditions for localization are given by eqn (6). However, in some instances $\mathbf{n}\cdot\mathbb{H}\cdot\mathbf{n}$ can be negative semi-definite at the beginning of the inelastic loading process and, due to relation (8), localization occurs immediately (continuous, discontinuous or both). Therefore eqn (8) will be the basis of the analysis herein.

The parameter G in eqn (2) is assumed to be strictly positive (i.e. g is greater than the snap-back threshold modulus g_s) and generally decreasing for a given loading process. The critical conditions for localization are looked for in terms of these parameters, mostly in terms of g. When localization does not occur immediately, eqn (6) furnishes, for a given $\mathbf{n}$, the value of g for which localization is possible in this direction. In order to find when eqn (8) is first satisfied during a loading process, one has to find the maximum of g for varying $\mathbf{n}$, i.e. one has to solve an optimization problem. In the following, an alternative procedure to solve this problem for a wide class of material models is presented and discussed. The general results will be presented in terms of H and h and are easily transformed in terms of G and g.

3. CRITICAL CONDITIONS AT THE INCEPTION OF LOCALIZATION

3.1. *Geometric interpretation of the localization condition*

Consider the tangent modulus in the loading condition defined by eqn (5):

$$\mathbb{H} = \mathbb{E} - \frac{(\mathbb{a}+p\mathbb{1})\otimes(\mathbb{b}+q\mathbb{1})}{H}. \tag{9}$$

The corresponding acoustic tensor is:

$$\mathbf{n}\cdot\mathbb{H}\cdot\mathbf{n} = \mu\mathbb{1} + \left(\lambda+\mu-\frac{pq}{H}\right)\mathbf{n}\otimes\mathbf{n} - \frac{1}{H}(\mathbf{n}\cdot\mathbb{a}\otimes\mathbb{b}\cdot\mathbf{n}+p\mathbf{n}\otimes\mathbb{b}\cdot\mathbf{n}+q\mathbf{n}\cdot\mathbb{a}\otimes\mathbf{n}), \tag{10}$$

where λ and μ are Lamé's constants, possibly affected by damage.

The determinant of the acoustic tensor is a bilinear symmetric form of $\mathbb{a}$ and $\mathbb{b}$ [see Rice (1976)]. This form can be transformed into the difference of two quadratic forms of $(\mathbb{a}+\mathbb{b},\ p+q)$ and $(\mathbb{a}-\mathbb{b},\ p-q)$, so that the localization condition can be written in the form:

$$\begin{aligned} H\mu(\lambda+2\mu) \leqslant {} & \frac{\lambda+2\mu}{4}[((\mathbb{a}+\mathbb{b})\cdot\mathbf{n})\cdot((\mathbb{a}+\mathbb{b})\cdot\mathbf{n})-((\mathbb{a}-\mathbb{b})\cdot\mathbf{n})\cdot((\mathbb{a}-\mathbb{b})\cdot\mathbf{n})] \\ & + \frac{\lambda+\mu}{4}[(\mathbf{n}\cdot(\mathbb{a}+\mathbb{b})\cdot\mathbf{n})^2-(\mathbf{n}\cdot(\mathbb{a}-\mathbb{b})\cdot\mathbf{n})^2]+\mu\frac{p+q}{2}\mathbf{n}\cdot(\mathbb{a}+\mathbb{b})\cdot\mathbf{n} \\ & - \mu\frac{p-q}{2}\mathbf{n}\cdot(\mathbb{a}-\mathbb{b})\cdot\mathbf{n}+\mu pq. \end{aligned} \tag{11}$$

For notation convenience we introduce two deviatoric tensors $\mathbb{N}$ and $\mathbb{M}$ and two scalars ρ and $\bar{\rho}$ defined as:

$$\mathbb{N} \equiv \mathbb{a}+\mathbb{b}, \quad \mathbb{M} \equiv \mathbb{a}-\mathbb{b}, \quad \rho \equiv p+q, \quad \bar{\rho} \equiv p-q. \tag{12}$$

It should be noted that, since $\mathbb{a}$ and $\mathbb{b}$ are unit tensors, the tensor $\mathbb{M}$ can be interpreted as a measure of the deviation from normality of inelastic strains in the deviatoric stress plane (deviatoric non-associativity), while $\bar{\rho}$ is a measure of the deviation from normality in a meridian stress plane (volumetric non-associativity).

Let us define the normal components Σ and $\bar{\Sigma}$ and the tangential components S and $\bar{S}$

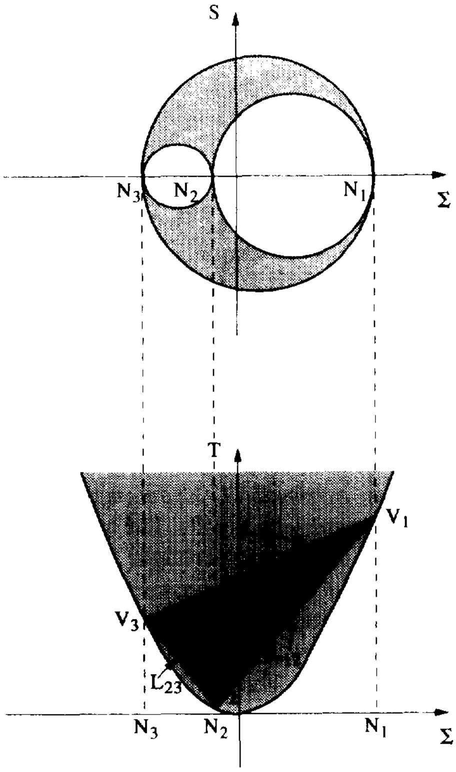

Fig. 1. Geometrical interpretation of the localization condition in the modified Mohr plane (Σ, T) and correspondence with the Mohr plane (Σ, S) for model (9).

of the vectors $\mathbb{N}\,.\,\mathbf{n}$ and $\mathbb{M}\,.\,\mathbf{n}$:

$$\Sigma = \mathbf{n}\,.\,\mathbb{N}\,.\,\mathbf{n}, \quad \bar{\Sigma} = \mathbf{n}\,.\,\mathbb{M}\,.\,\mathbf{n},$$

$$S = [(\mathbb{N}\,.\,\mathbf{n})\,.\,(\mathbb{N}\,.\,\mathbf{n}) - (\mathbf{n}\,.\,\mathbb{N}\,.\,\mathbf{n})^2]^{1/2}, \quad \bar{S} = [(\mathbb{M}\,.\,\mathbf{n})\,.\,(\mathbb{M}\,.\,\mathbf{n}) - (\mathbf{n}\,.\,\mathbb{M}\,.\,\mathbf{n})^2]^{1/2}. \tag{13}$$

For varying normal $\mathbf{n}$, Σ, S (or $\bar{\Sigma}$, $\bar{S}$) describe in the Mohr plane the region delimited by the three Mohr circles (shaded region in Fig. 1). An alternative representation, which turns out to be more convenient for our purposes, is obtained by introducing the square magnitudes T and $\bar{T}$ of the vectors $\mathbb{N}\,.\,\mathbf{n}$ and $\mathbb{M}\,.\,\mathbf{n}$, respectively:

$$T = S^2 + \Sigma^2, \quad \bar{T} = \bar{S}^2 + \bar{\Sigma}^2. \tag{14}$$

The Mohr plane (Σ, S) (or $\bar{\Sigma}$, $\bar{S}$) transforms into the region $T \geqslant \Sigma^2$ (or $\bar{T} \geqslant \Sigma^2$) of the (Σ, T) (or $\bar{\Sigma}$, $\bar{T}$) plane and the admissible Mohr region becomes a triangle (see Fig. 1).

Introducing definitions (13) and (14) into the localization conditions (11) one obtains:

$$(\lambda+2\mu)T-(\lambda+\mu)\Sigma^2+2\mu\Sigma\rho+\mu\rho^2 \geqslant 4\mu(\lambda+2\mu)H+(\lambda+2\mu)\bar{T}-(\lambda+\mu)\bar{\Sigma}^2+2\mu\bar{\Sigma}\bar{\rho}+\mu\bar{\rho}^2. \tag{15}$$

We assume that at the beginning of the inelastic process the inequality (15) is not fulfilled, i.e. we assume that localization does not occur immediately outside the elastic range, and that g decreases along the loading history. In this case the inception of localization corresponds to the equality sign in eqn (15). This assumption will be removed in Section 3.4.

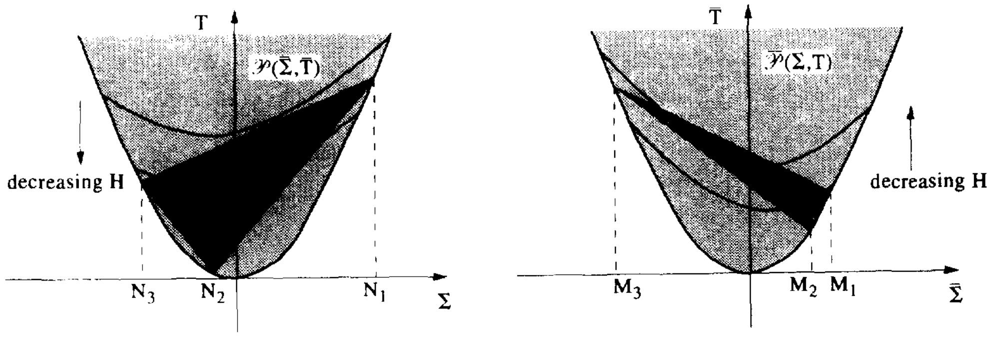

Fig. 2. Geometrical interpretation of the localization condition in the (Σ, T) and $(\bar{\Sigma}, \bar{T})$ planes for model (9).

Equation (15) with the equality sign can be regarded either as the equation of a parabola with vertical axis $\mathscr{P}$ in the plane (T, Σ) or as the equation of another parabola with vertical axis $\bar{\mathscr{P}}$ in the $(\bar{T}, \bar{\Sigma})$ plane (see Fig. 2). The positions of these parabolas depend on the normal $\mathbf{n}$ through $\bar{T}$ and $\bar{\Sigma}$ for $\mathscr{P}$ and through T and Σ for $\bar{\mathscr{P}}$. For decreasing H, $\mathscr{P}$ translates down in the T direction and $\bar{\mathscr{P}}$ translates up. This geometric interpretation furnishes a convenient tool to solve localization conditions explicitly.

Consider a normal $\mathbf{n}^*$: to this normal corresponds a point (T^*, Σ^*) in the (T, Σ) plane and a point $(\bar{T}^*, \bar{\Sigma}^*)$ in the $(\bar{T}, \bar{\Sigma})$ plane. There will be a solution to the localization problem in the direction $\mathbf{n}^*$ if the parabola $\mathscr{P}^*$ corresponding to this direction, i.e. evaluated with $\bar{T} = \bar{T}^*$ and $\bar{\Sigma} = \bar{\Sigma}^*$, intersects the admissible area of the (T, Σ) plane and the point (T^*, Σ^*) belongs to this intersection (see Fig. 3).

3.2. *Critical hardening moduli and critical normals to localization planes—potential onsets of localization*

The analysis greatly simplifies if $\mathbb{N}$ and $\mathbb{M}$ are coaxial, i.e. if α and β are coaxial. With this hypothesis, which will be assumed in the following, one can work only in one plane, e.g. the plane (T, Σ); compute $\bar{T}$ and $\bar{\Sigma}$ as functions of T and Σ and put these expressions in relation (15), ending with an expression involving only T and Σ. To this purpose we make use of the classical formulae of Mohr analysis, defining the direction $\mathbf{n}$ corresponding to a given point T, Σ belonging to the dashed area of the (T, Σ) plane, by its components n_i *in the principal frame of* $\mathbb{N}$ as:

$$\begin{aligned} T + N_1\Sigma + N_2N_3 &= n_1^2(N_1 - N_2)(N_1 - N_3) \\ T + N_2\Sigma - N_1N_3 &= n_2^2(N_2 - N_1)(N_2 - N_3) \\ T + N_3\Sigma - N_1N_2 &= n_3^2(N_3 - N_1)(N_3 - N_2), \end{aligned} \tag{16}$$

where N_1, N_2 and N_3 are the eigenvalues of the tensor $\mathbb{N}$. Let us also denote by M_1, M_2,

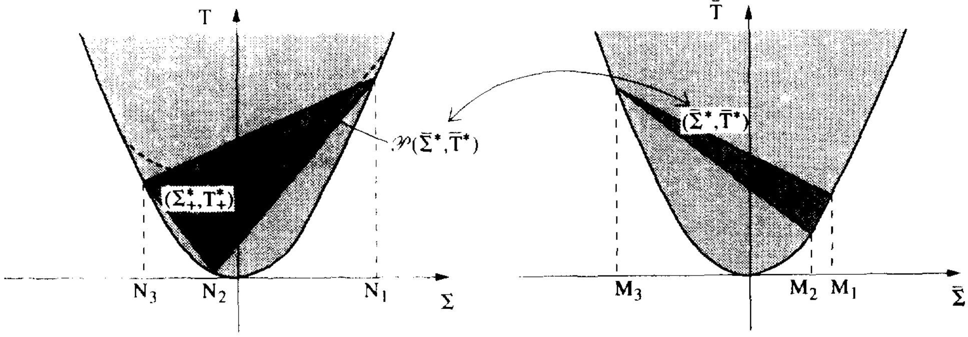

Fig. 3. Relation between the critical conditions in the (Σ, T) and $(\bar{\Sigma}, \bar{T})$ planes.

M_3 the eigenvalues of the tensor $\mathbb{M}$. Due to the assumed coaxiality of $\mathbb{N}$ and $\mathbb{M}$, we can easily compute $\bar{T}$ and $\bar{\Sigma}$ as:

$$\bar{\Sigma} = M_1 n_1^2 + M_2 n_2^2 + M_3 n_3^2 \tag{17}$$

$$\bar{T} = M_1^2 n_1^2 + M_2^2 n_2^2 + M_3^2 n_3^2. \tag{18}$$

In order to find explicitly the critical values of H (or h) one has to express the localization condition in the (T, Σ) plane or in the $(\bar{T}, \bar{\Sigma})$ plane only. Different cases must be treated separately.

(1) At least one tensor $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ does not exhibit any symmetry (i.e. it has three distinct eigenvalues) or both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are axially symmetric but the axis of symmetry of $\boldsymbol{\alpha}$ (say $\mathbf{x}_i$) is different from the axis of symmetry of $\boldsymbol{\beta}$ (say $\mathbf{x}_j$). In this situation one can always assume that $\mathbb{N}$ *has three distinct eigenvalues*, since if this is not the case it is possible to replace, in the tangent modulus $\mathbb{H}$, $\boldsymbol{\beta}$ by $\boldsymbol{\beta}' = c\boldsymbol{\beta}$ and G by $G' = cG$, where c is a non-vanishing scalar, such that $\boldsymbol{\alpha}+\boldsymbol{\beta}'$ has distinct eigenvalues.

(2) Tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are axially symmetric with the same axis of symmetry or one of them is axially symmetric and the other is spherical. In this case one can assume that $\mathbb{N}$ and $\mathbb{M}$ *each have a double eigenvalue corresponding to the same axes*. If $\mathbb{N}$ or $\mathbb{M}$ vanishes one can always find a scalar c such that neither $\boldsymbol{\alpha}+c\boldsymbol{\beta}$ nor $\boldsymbol{\alpha}-c\boldsymbol{\beta}$ is spherical.

(3) Tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are spherical ($\mathbb{a} = \mathbb{b} = \mathbb{O}$). From eqn (11) it follows directly that the critical value of H is:

$$H = \frac{pq}{\lambda+2\mu} \tag{19}$$

and the normal to the localization plane is arbitrary.

(1) $\mathbb{N}$ *has three distinct eigenvalues*. The N_i eigenvalues being distinct, using formulae (16) it is easy to derive the expressions of the components n_i^2 ($i = 1, 2, 3$). Substituting these expressions in eqns (17) and (18) and the results obtained in eqn (15), the localization condition in the plane (T, Σ) reads:

$$\begin{aligned}
\mathscr{F}(T,\Sigma,H) &= (\lambda+2\mu)T-(\lambda+\mu)\Sigma^2+2\mu\Sigma\rho+\mu\rho^2-4\mu(\lambda+2\mu)H \\
&-(\lambda+2\mu)\left[M_1^2\frac{T+N_1\Sigma+N_2N_3}{(N_1-N_2)(N_1-N_3)}+M_2^2\frac{T+N_2\Sigma+N_1N_3}{(N_2-N_1)(N_2-N_3)}+M_3^2\frac{T+N_3\Sigma+N_1N_2}{(N_3-N_1)(N_3-N_2)}\right] \\
&+(\lambda+\mu)\left[M_1\frac{T+N_1\Sigma+N_2N_3}{(N_1-N_2)(N_1-N_3)}+M_2\frac{T+N_2\Sigma+N_1N_3}{(N_2-N_1)(N_2-N_3)}+M_3\frac{T+N_3\Sigma+N_1N_2}{(N_3-N_1)(N_3-N_2)}\right]^2 \\
&-2\mu\left[M_1\frac{T+N_1\Sigma+N_2N_3}{(N_1-N_2)(N_1-N_3)}+M_2\frac{T+N_2\Sigma+N_1N_3}{(N_2-N_1)(N_2-N_3)}+M_3\frac{T+N_3\Sigma+N_1N_2}{(N_3-N_1)(N_3-N_2)}\right]\bar{\rho} \\
&-\mu\bar{\rho}^2 = 0.
\end{aligned} \tag{20}$$

Equation (20) represents in general a conical curve, changing in position and/or in "size" with H. $\mathscr{F}$ can be expressed in the following form:

$$\mathscr{F}(T,\Sigma;H) = a\Sigma^2-bT^2+c\Sigma T+d\Sigma+eT+f(H) = 0. \tag{21}$$

The coefficients a–e and the function f, obtained from eqn (20), are reported in Appendix B. It is also shown in the Appendix that $\mathscr{F}$ represents either a hyperbola or a parabola with

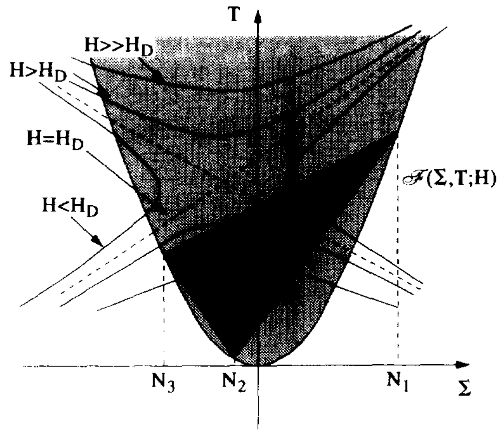

Fig. 5. Sketch of various possibilities of localization in the (Σ, T) plane.

a vertical axis. This last situation occurs, for instance, if the material model is associative on the deviatoric part ($\mathbb{M} = \mathbb{O}$). The hyperbola (21) degenerates into two straight lines when

$$f(H) = -4\mu(\lambda+2\mu)H = \frac{-e^2a+d^2b-ecd}{c^2-4ab}.$$

This equation gives a characteristic value

$$H_D = -\frac{1}{4\mu(\lambda+2\mu)}\frac{-e^2a+d^2b-ecd}{c^2-4ab}. \tag{22}$$

In the general case the hyperbola has two asymptotes which are fixed for varying H; on the contrary, its focal distance changes with H. Figure 4 shows its different shapes for different values of H.

Given a loading process, inception of localization will depend on the relative position of this curve with respect to the triangular dashed area at the beginning of the inelastic process. When localization does not occur immediately it can occur *a priori* in six different ways: either the conical curve intersects the dashed area first at one of its vertices or it becomes tangential to any of the line segments defining this area, as sketched in Fig. 5. Depending on the constitutive behaviour at hand, some or all of these possibilities may be ruled out. We will come back to this point later on. Evaluation of the critical conditions is now easy.

(i) In the case where localization occurs by touching a vertex V_k ($k = 1, 2, 3$), we have $\Sigma = N_k$ and $T = N_k^2$ and the critical value of H at localization is easily derived from relation (20) by substituting these values to obtain:

$$H_k = \frac{\alpha_k\beta_k}{\lambda+2\mu}. \tag{23}$$

The corresponding normal to the localization plane is simply the kth principal direction of $\mathbb{N}$, i.e.

$$n_k = 1, \quad n_j = 0 \text{ if } j \neq k. \tag{24}$$

The modulus h for the model considered is related to H by the equation:

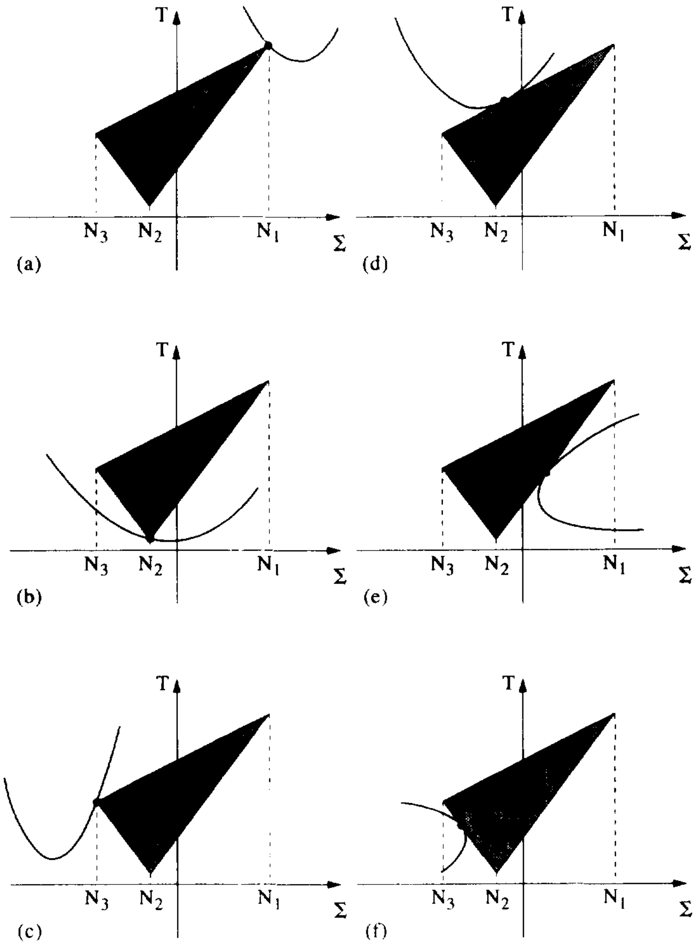

Fig. 4. Geometrical interpretation of the localization condition in the (Σ, T) plane: evolution with H of the curve representing continuous localization.

$$h = H - \frac{(N_1^2+N_2^2+N_3^2)-(M_1^2+M_2^2+M_3^2)}{8\mu} - \frac{3}{4(3\lambda+2\mu)}[\rho^2-\bar{\rho}^2]. \tag{25}$$

If $(N_i - N_j)^2 \neq (M_i - M_j)^2$ [the triplet (i, j, k) taking the values (1, 2, 3), (2, 3, 1) or (3, 1, 2)] the modulus h_k corresponding to H_k can be written as an algebraic sum of squares:

$$h_k = \left\{ -\frac{4\mu}{(\lambda+2\mu)(3\lambda+2\mu)} \left[\frac{3\lambda+2\mu}{4\mu}(N_k(N_i-N_j) - M_k(M_i-M_j)) - \rho(N_i-N_j) \right.\right.$$
$$\left. + \bar{\rho}(M_i-M_j) \right]^2 + \frac{4\mu}{(\lambda+2\mu)(3\lambda+2\mu)} \left[\frac{3\lambda+2\mu}{4\mu}(N_k(M_i-M_j) - M_k(N_i-N_j)) - \rho(M_i-M_j) \right.$$
$$\left.\left. + \bar{\rho}(N_i-N_j) \right]^2 - \frac{1}{4\mu}\cdot[(N_i-N_j)^2-(M_i-M_j)^2]^2 \right\} \frac{1}{(N_i-N_j)^2-(M_i-M_j)^2}. \tag{26}$$

(ii) When localization occurs by tangency to the line segment L_{ij} joining the vertices V_i and V_j and whose equation is [here again the triplet (i, j, k) assumes the values (1, 2, 3), (2, 3, 1) or (3, 1, 2)]

$$T+N_iN_j+N_k\Sigma = 0, \quad T \geqslant \Sigma^2, \tag{27}$$

the tangency condition (obtained by writing the orthogonality of the normal to the conic to L_{ij}) is:

$$\frac{\partial \mathscr{F}}{\partial T} N_k = \frac{\partial \mathscr{F}}{\partial \Sigma}. \tag{28}$$

Using eqn (20), relation (26) gives:

$$2(\lambda+\mu)\Sigma A_{ij} = \{-(\lambda+2\mu)N_k(N_i-N_j)^2+(\lambda+\mu)N_k(M_i-M_j)^2+\mu M_k(N_i-N_j)(M_i-M_j)$$
$$-2\mu\bar{\rho}(N_i-N_j)(M_i-M_j)+2\mu\rho(N_i-N_j)^2\}, \tag{29}$$

where we have set

$$A_{ij} \equiv (N_i-N_j)^2-(M_i-M_j)^2. \tag{30}$$

Provided A_{ij} is non-zero, and Σ is between N_i and N_j, eqn (29) furnishes the value of Σ at the tangency point,

$$\Sigma = -\frac{N_k}{2}+\frac{\mu(N_i-N_j)\{(M_k-2\bar{\rho})(M_i-M_j)-(N_k-2\rho)(N_i-N_j)\}}{2(\lambda+\mu)A_{ij}}, \tag{31}$$

while eqn (27) gives the value of T:

$$T = -N_k\Sigma-N_iN_j. \tag{32}$$

Substituting these values of Σ and T in the localization condition (20) yields the critical value of H at localization H_{ij},

$$H_{ij} = \frac{1}{16\mu(\lambda+\mu)A_{ij}}\Big\{\mu[(M_k-2\bar{\rho})(M_i-M_j)-(N_k-2\rho)(N_i-N_j)]^2$$
$$+(\lambda+\mu)[(N_i-N_j)^2-(M_i-M_j)^2]^2-\frac{\mu(\lambda+\mu)}{\lambda+2\mu}[(N_k-2\rho)(M_i-M_j)$$
$$-(M_k-2\rho)(N_i-N_j)]^2\Big\}. \tag{33}$$

The modulus h_{ij} corresponding to H_{ij} [eqn (33)] can be written in the form:

$$h_{ij} = \frac{1}{4A_{ij}}\Big\{-\frac{\mu}{(3\lambda+2\mu)(\lambda+\mu)}\Big[\frac{3\lambda+2\mu}{2\mu}(N_k(N_i-N_j)-M_k(M_i-M_j))$$
$$+\rho(N_i-N_j)-\bar{\rho}(M_i-M_j)\Big]^2+\frac{4\mu}{(\lambda+2\mu)(3\lambda+2\mu)}\Big[\frac{3\lambda+2\mu}{8\mu}(N_k(M_i$$
$$-M_j)-M_k(N_i-N_j))+\rho(M_i-M_j)-\bar{\rho}(N_i-N_j)\Big]^2$$
$$+\frac{9}{16\mu}[N_k(M_i-M_j)-M_k(N_i-N_j)]^2\Big\}. \tag{34}$$

When the critical hardening modulus is $h_{i,j}$, the components of the normal to the localization plane are obtained from eqn (16) by setting $n_k = 0$ and using the value of Σ at the tangency point [eqn (31)]

$$n_i^2 = \frac{\Sigma - N_j}{N_i - N_j} = \frac{1}{2} + \frac{\mu\{(M_k - 2\bar{\rho})(M_i - M_j) - (N_k - 2\rho)(N_i - N_j)\}}{2(\lambda+\mu)A_{ij}} \tag{35}$$

$$n_j^2 = -\frac{\Sigma - N_i}{N_i - N_j} = \frac{1}{2} - \frac{\mu\{(M_k - 2\bar{\rho})(M_i - M_j) - (N_k - 2\rho)(N_i - N_j)\}}{2(\lambda+\mu)A_{ij}}. \tag{36}$$

If $A_{ij} = (N_i - N_j)^2 - (M_i - M_j)^2 = 0$, two possibilities arise: (a) the right-hand side of eqn (29) is non-zero—localization cannot occur by tangency of the hyperbola to the side L_{ij} of the triangle; (b) the right-hand side of eqn (29) is zero. The tangency occurs along all the i–j side of the triangle including the vertices V_i and V_j; the critical value of H is determined by the condition which makes the hyperbola degenerate into two straight lines, i.e. H_D defined above. One of these lines is the side L_{ij} of the triangle. Moreover, in this case $H_D = H_i = H_j$, H_i and H_j defined by eqn (23).

(2) $\mathbb{M}$ *and* $\mathbb{N}$ *each have a double eigenvalue*. We denote by j and k the axes corresponding to the equal eigenvalues and by i the axis corresponding to the simple eigenvalue of $\mathbb{N}$ and $\mathbb{M}$. Therefore $N_j = N_k = -N_i/2$ and $M_j = M_k = -M_i/2$. Relations (17) and (18) simplify to:

$$\bar{\Sigma} = \frac{M_i}{2}(3n_i^2 - 1), \quad \bar{T} = \frac{M_i^2}{4}(3n_i^2 + 1). \tag{37}$$

n_i can be expressed as a function of Σ_+ and T, by eqn (16):

$$n_i^2 = \frac{4T}{9N_i^2} + \frac{4\Sigma}{9N_i} + \frac{1}{9}. \tag{38}$$

Substitution of eqns (37) and (38) in the localization condition (15) gives:

$$\tfrac{4}{9}(\lambda+\mu)\frac{M_i^2}{N_i^4}T^2 + (\lambda+\mu)\cdot\left(\frac{4}{9}\frac{M_i^2}{N_i^2} - 1\right)\Sigma^2 + \tfrac{8}{9}(\lambda+\mu)\frac{M_i^2}{N_i^3}\Sigma T$$
$$+\left[\lambda + 2\mu - \frac{7\lambda+10\mu}{9}\frac{M_i^2}{N_i^2} - \tfrac{4}{3}\mu\frac{M_i}{N_i^2}\bar{\rho}\right]T + \left[2\mu\rho - \frac{7\lambda+10\mu}{9}\frac{M_i^2}{N_i} - \tfrac{4}{3}\mu\frac{M_i}{N_i}\bar{\rho}\right]\Sigma + \mu(\rho^2 - \bar{\rho}^2)$$
$$-\frac{2\lambda+5\mu}{9}M_i^2 + \tfrac{2}{3}\mu\bar{\rho}M_i - 4\mu(\lambda+2\mu)H = 0. \tag{39}$$

The above equation again represents a hyperbola in the (T, Σ) plane. If localization does not occur immediately, the critical value of H is found by imposing the tangency of the hyperbola to the transformed Mohr diagram, which in this case reduces to the segment $L_{ik} \equiv L_{ij}$, or by writing that the hyperbola passes through the vertices $V_k \equiv V_j$ or V_i. In this last case, $\Sigma = N_i$, $T = N_i^2$ (or $\Sigma = N_k$, $T = N_k^2$) and we obtain:

$$H_p = \frac{\alpha_p\beta_p}{(\lambda+2\mu)} \quad \text{for}\, p = (i, j, k). \tag{40}$$

The set of corresponding normals to the localization plane is the plane Π_{jk} spanned by the axes associated with the equal eigenvalues [when $p = j$ or k in eqn (40)] or the direction associated with the single eigenvalue [when $p = i$ in eqn (40)].

It is interesting to note here that although the reasoning used was somewhat different from that used in the case of distinct eigenvalues, the critical values of H given by eqns (40) and (23) coincide. The differences lie only on the normals to the localization plane. One may also check that this holds when localization occurs by tangency to the segment $L_{ik} \equiv L_{ij}$, and the critical value of H at localization is still given by eqn (33).

The normals to the localization planes belong to the cone Γ_i with the axis direction $\mathbf{n}_i$ and with angle θ defined by:

$$\cos^2\theta = n_i^2 = \frac{3\lambda+4\mu}{6} + \frac{2\mu}{3(\lambda+\mu)}\frac{N_i\rho - M_i\bar{\rho}}{N_i^2 - M_i^2}. \tag{41}$$

3.3. *Critical hardening moduli and their domain of validity*

To find the real critical value of H (or h) at the inception of localization one has to compute the maximum between the "admissible" H_{i-j} given by eqn (23) and the H_k given by eqn (23) for i, j and $k \in \{1, 2, 3\}$. By "admissible H_{i-j}" we mean here the values of H_{i-j} [eqn (33)] such that the corresponding value of Σ at the tangency point [eqn (31)] belongs to the segment L_{ij} [eqn (27)]. This condition may be written in the form:

$$-1 \leqslant \frac{\mu}{(\lambda+\mu)A_{ij}}\{(M_k-2\bar{\rho})(M_i-M_j)-(N_k-2\rho)(N_i-N_j)\} \leqslant 1. \tag{42}$$

For each couple (i, j), relation (42) is better sketched in the plane $(\rho, \bar{\rho})$, where equalities correspond to two straight and parallel lines ℓ_{ij} and $\bar{\ell}_{ij}$ (see Fig. 6).

Using eqns (23) and (33), one can compute the differences:

$$H_{i-j} - H_i = \frac{(\lambda+\mu)A_{ij}}{16\mu(\lambda+2\mu)}B_{ij}^2 \tag{43}$$

$$H_{i-j} - H_j = \frac{(\lambda+\mu)A_{ij}}{16\mu(\lambda+2\mu)}B_{ji}^2 \tag{44}$$

$$H_i - H_j = \frac{(M_k-2\bar{\rho})(M_i-M_j)-(N_k-2\rho)(N_i-N_j)}{4(\lambda+2\mu)}, \tag{45}$$

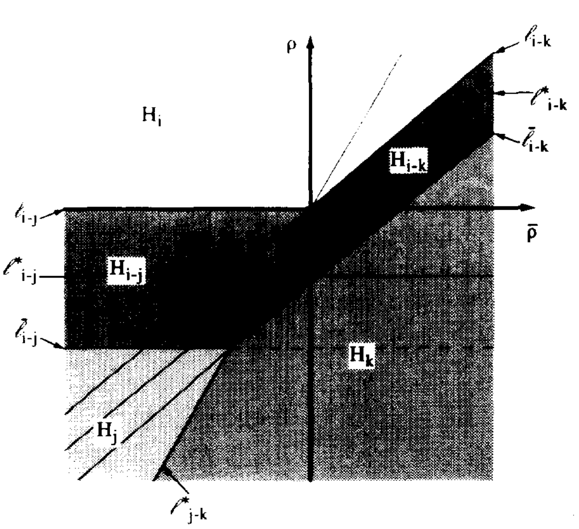

Fig. 6. Domains of validity of h_i and h_{i-j} in the $(\rho, \bar{\rho})$ plane for $N_i = -N_k$, $N_j = 0$, $M_i = M_j = -M_k/2$.

where

$$B_{ij} = \frac{\mu}{(\lambda+\mu)A_{ij}}\{(M_k-2\bar{\rho})(M_i-M_j)-(N_k-2\rho)(N_i-N_j)\}-1. \tag{46}$$

If $A_{ij} > 0$, relations (43) and (44) show that H_{i-j} is always greater than H_i and H_j but, in view of eqn (42), it is valid only between the lines ℓ_{ij} and $\bar{\ell}_{ij}$ while H_i and H_j hold outside. Further, relation (45) shows that H is valid on one side and H_j on the other side of the domain delimited by these two lines, as this relation is represented in the same diagram by a line l^*_{ij} which is parallel to ℓ_{ij} and $\bar{\ell}_{ij}$ and interior to the strip ℓ_{ij}–$\bar{\ell}_{ij}$. If $A_{i,j} < 0$, from the same relations one may conclude that the critical hardening modulus is never H_{i-j}.

We have seen up to now that for each side of the triangle defined by the vertices V_i and V_j, three or two expressions are available for the critical conditions. For each side, we have defined the domain of validity of these expressions. It remains now to see which side is actually concerned by first occurrence of localization. To this purpose, consider $A_{ij} > 0$ and compute from eqns (43) and (45) the differences:

$$H_{i-j}-H_{i-k} = (H_{i-j}-H_i)-(H_{i-k}-H_i) = \frac{\lambda+\mu}{16\mu(\lambda+2\mu)}(A_{ij}B_{ij}^2-A_{ik}B_{ik}^2) \tag{47}$$

$$H_{i-j}-H_k = (H_{i-j}-H_i)-(H_k-H_i) = \frac{\lambda+\mu}{16\mu(\lambda+2\mu)}[A_{ij}B_{ij}^2+4A_{ik}(B_{ik}+1)]. \tag{48}$$

Note that if $A_{ik} > 0$, the condition $H_{i-j} > H_{i-k}$ also implies $H_{i-j} > H_k$ [compare eqn (48)], while if $A_{ik} < 0$, H_{i-j} is always greater than H_{i-k} but may be smaller than H_k. Reassembling all these results one may conclude the following, depending on the sign of A_{ij}, A_{ik}, A_{jk} with the triplet (i, j, k), assuming the values (1, 2, 3), (2, 3, 1) or (3, 1, 2).

(a) If $A_{ij} > 0$ for any i, j distinct and belonging to the set (1, 2, 3), the critical hardening modulus is H_{i-j} if and only if:

$$B_{ij} \leqslant 0, \quad B_{ji} \leqslant 0, \quad A_{ij}B_{ij}^2 \geqslant A_{ik}B_{ik}^2, \quad A_{ji}B_{ji}^2 \geqslant A_{jk}B_{jk}^2, \tag{49}$$

while it is H_i if and only if:

$$B_{ij} \geqslant 0, \quad B_{ik} \geqslant 0. \tag{50}$$

(b) If $A_{ij} > 0$, $A_{ik} > 0$ while $A_{jk} < 0$, the critical hardening modulus is H_{i-j} if and only if:

$$B_{ij} \leqslant 0, \quad B_{ji} \leqslant 0, \quad A_{ij}B_{ij}^2 \geqslant A_{ik}B_{ik}^2; \tag{51}$$

it is H_{i-k} if and only if:

$$B_{ik} \leqslant 0, \quad B_{ki} \leqslant 0, \quad A_{ik}B_{ik}^2 \geqslant A_{ij}B_{ij}^2. \tag{52}$$

while it is H_i if and only if:

$$B_{ij} \geqslant 0, \quad B_{ik} \geqslant 0. \tag{53}$$

Finally, it is H_j (H_k) if and only if:

$$B_{ji} \geqslant 0, \quad B_{jk} \leqslant -1 \; (B_{ki} \geqslant 0, B_{kj} \leqslant -1). \tag{54}$$

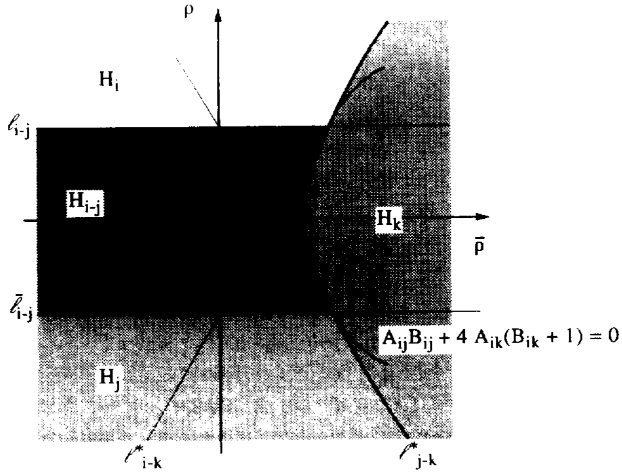

Fig. 7. Domains of validity of h_i and h_{i-j} in the $(\rho, \bar{\rho})$ plane for $N_i = -N_j, N_k = 0, M_i = M_j = -M_k/2$.

In Fig. 6, the regions of validity of each expression of H_c are represented in the plane $(\rho, \bar{\rho})$ for the particular case $N_i = -N_k, N_j = 0, M_i = M_j = -M_k/2$.

(c) If $A_{ij} > 0$, while $A_{ik} < 0$ and $A_{jk} < 0$, the critical hardening modulus is H_{i-j} if and only if:

$$B_{ij} \leqslant 0, \quad B_{ji} \leqslant 0, \quad A_{ij}B_{ij}^2 + 4A_{ik}(B_{ik}+1) \geqslant 0\,; \tag{55}$$

it is H_i (or H_j) if and only if:

$$B_{ij} \geqslant 0, \quad B_{ik} \leqslant -1 \quad (B_{ij} \geqslant 0, B_{jk} \leqslant -1), \tag{56}$$

while it is H_k if and only if:

$$B_{ki} \leqslant -1, \quad B_{kj} \leqslant -1, \quad A_{ij}B_{ij}^2 + 4A_{ik}(B_{ik}+1) \leqslant 0. \tag{57}$$

In Fig. 7, these different regions are shown again in the $(\rho, \bar{\rho})$ plane for the particular case $N_i = -N_j, N_k = 0, M_i = M_j = -M_k/2$.

(d) If $A_{ij} < 0$ for any i, j distinct and belonging to the set (1, 2, 3), the critical hardening modulus is H_i if and only if:

$$B_{ij} \leqslant -1, \quad B_{ik} \leqslant -1. \tag{58}$$

Remark 1. In Bigoni and Hueckel (1991), closed form solutions of the localization problem are achieved by the Lagrange multipliers method for the same class of material models considered here. The geometrical method proposed here allows one to obtain more precise results. First, the critical conditions are given more explicit forms. Then, the validities of the various expressions of these critical conditions are also defined completely. Further, it has been shown that the normal to a critical localization plane is, except in the particular case where $\mathbb{a}$ and $\mathbb{b}$ each have a double eigenvalue, always orthogonal to a principal plane of $\mathbb{a}$ or $\mathbb{b}$. Therefore, the solution proposed by Bigoni and Hueckel, not satisfying this condition (see case 1) in Section 3 of their paper, should be excluded *a priori*. The obtained results are also in agreement with those of Ottosen and Runesson (1991b) relative to a non-associated Mohr material.

Remark 2. Equations (34) and (26) give the critical hardening modulus h_{i-j} or h_k as algebraic sums of squares. This particular form shows that, as expected, the critical hardening modulus is never positive for associative constitutive behaviour. In this case, indeed, $\mathbb{M} = \mathbb{O}$ and $\bar{\rho} = 0$, therefore the last two terms of h_{i-j} and the second term in h_k

vanish. The same expressions of h_{i-j} and h_k also highlight the roles of non-associativity, be it deviatoric ($\mathbb{M} \neq \mathbb{O}$, $\bar{\rho} = 0$), hydrostatic ($\mathbb{M} = \mathbb{O}$, $\bar{\rho} \neq 0$) or a combination of both ($\mathbb{M} \neq \mathbb{O}$ and $\bar{\rho} \neq 0$). Any of these non-associativities may result in localization in the hardening ($h > 0$) regime.

Remark 3. As noted already, for particular material models some of the various possibilities of localization sketched in Fig. 5 are *a priori* excluded. For instance, if the model exhibits deviatoric associativity, i.e. if $\mathbb{a} = \mathbb{b}$ ($\mathbb{M} = \mathbb{O}$), only one of the h_{i-j} [eqn (34)] and two of the h_k [eqn (26)] need to be considered. In fact, in this case the localization condition (15) turns out to depend on T and Σ only and (with the equality sign) represents a parabola with vertical axis which translates down in the T direction for decreasing H. Denoting by N_1 and N_3, respectively, the maximum and the minimum eigenvalues of $\mathbb{N}$, localization can occur only when the parabola passes through the vertices V_1 or V_3 (in this case the critical hardening modulus is h_1 or h_3) or becomes tangential to the segment L_{13} (in this case the critical hardening modulus is h_{1-3}). The Druker–Prager material model with different values of the friction and dilatancy angles already treated by Rudnicki and Rice (1975) and by Perrin and Leblond (1994) belongs to this class of models. Also, some isotropic damage behaviours can be described by a tangent tensor of the form (9) with $\mathbb{a} = \mathbb{b}$. An example will be discussed in Section 4.

3.4. *Continuous versus discontinuous localization at the beginning of the inelastic process*

Consider again the necessary and sufficient condition of localization for a real solid expressed by the inequality (15). It is possible to determine the sets of initial inelastic states for which localization occurs immediately at the beginning of the inelastic process. In this case discontinuous localization [corresponding to the strict inequality in eqn (15)] can occur simultaneously, or even precede, continuous localization [corresponding to the equality in eqn (15)].

For simplicity, in this section the discussion is restricted to deviatoric associative models, i.e. to $\mathbb{a} = \mathbb{b}$, $\mathbb{M} = \mathbb{O}$ (see Remark 3). Then condition (15) represents in T, Σ space the epigraph of a parabola with vertical axis $\mathscr{P}$. Three different situations may occur, depending on the relative position of the parabola $\mathscr{P}^0$ and of the admissible Mohr region (triangle $\mathscr{T}^0$) at the beginning of the inelastic process (superscript 0); they are shown in Fig. 8.

(a) The parabola is completely above the triangle. Localization is not possible for $h = h_0$. Bifurcation in the form of strain rate discontinuity can occur later, during the inelastic process, and continuous localization will precede discontinuous localization. This case has been treated in the previous sections. The corresponding set of initial inelastic states can be found by imposing that the difference between T^0 computed for the parabola and T^0 computed along the side L_{13} of the Mohr triangle be positive for each $N_3^0 \leqslant \Sigma^0 \leqslant N_1^0$:

$$\frac{\lambda+\mu}{\lambda+2\mu}\Sigma^2+\left(N_2^0-\frac{2\mu}{\lambda+2\mu}\rho^0\right)\Sigma-\frac{\mu}{\lambda-2\mu}((\rho^0)^2-(\bar{\rho}^0)^2)+4\mu H^0+N_1^0N_3^0>0. \quad (59)$$

Condition (59) is fulfilled if the discriminant of its left-hand side is negative, namely:

$$\left(N_2^0-\frac{2\mu}{\lambda+2\mu}\rho^0\right)^2-4\frac{\lambda+\mu}{\lambda+2\mu}\cdot\left(-\frac{\mu}{\lambda+2\mu}((\rho^0)^2-(\bar{\rho}^0)^2)+4\mu H^0+N_1^0N_3^0\right)<0. \quad (60)$$

(b) The parabola $\mathscr{P}^0$ intersects the admissible area $\mathscr{T}^0$. Localization occurs instantaneously at the beginning of the inelastic process at the critical hardening modulus h^0. The normals to the possible continuous localization planes are those corresponding to the

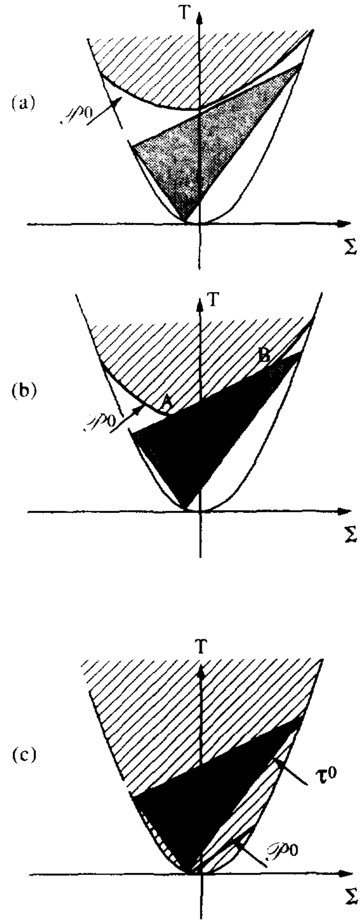

Fig. 8. Modified Mohr diagram $\mathscr{T}^0$ and epigraph of the parabola representing the localization condition at the beginning of the inelastic processes: (a) localization can occur only for $h < h^0$; (b) continuous and discontinuous localizations occur for $h = h^0$; and (c) discontinuous localization occurs for $h = h^0$.

arc AB in Fig. 8(b), while the normals to the possible discontinuous localization planes correspond to the dark shaded areas in the same figure.

(c) The parabola $\mathscr{P}^0$ does not intersect the modified Mohr diagram $\mathscr{T}^0$, but the epigraph of this parabola contains $\mathscr{T}^0$ [see Fig. 8(c)]. In this case relation (15) is satisfied by h^0 as a strict inequality for any Σ, T belonging to the Mohr diagram. Discontinuous localization is then immediately possible, the critical hardening modulus h^0 and the normal to the localization plane being indeterminate. It should be pointed out that in these particular situations discontinuous localization *precedes* the continuous one which can eventually occur later when the parabola reaches the admissible triangle.

Immediate localization for $h = h^0$ [cases (b) and (c)] is possible for all the initial inelastic states which do no not satisfy condition (60).

4. EXAMPLES

4.1. *Application to elastic–perfectly plastic and damageable materials*

The main goal of this simple example is to present a situation where localization can occur at the beginning of the inelastic process or only after a finite amount in this process, depending on the type of loading condition, and to identify the corresponding types of localization. The constitutive model for elastic–perfectly plastic and damageable materials is due to Lemaitre (1992) and has been used by Benallal *et al.* (1992) to analyse critical damage states at crack initiation. With this constitutive behaviour the tangent modulus is given by:

$$\mathbb{H} = (1-D)\mathbb{E} - \frac{\left\{\frac{E}{1+\nu}\left[\frac{3}{2\bar{\sigma}} + \frac{(1+\nu)\bar{\sigma}^2 R_\nu}{2ZE^2(1-D)^4}\right]\mathbb{s} + \frac{\bar{\sigma}^2 R_\nu \operatorname{tr}(\sigma)}{6ZE(1-D)^4}\mathbb{1}\right\} \otimes \frac{3E}{2(1+\nu)\bar{\sigma}}\mathbb{s}}{g + \frac{E}{(1+\nu)(1-D)}\left[\frac{3}{2\bar{\sigma}} + \frac{(1+\nu)\bar{\sigma}^2 R_\nu}{2ZE^2(1-D)^4}\right]\bar{\sigma}}, \tag{61}$$

where Z is a material parameter. E and ν are the elastic constants and R_ν is the triaxiality function defined as follows:

$$R_\nu = \tfrac{2}{3}(1-\nu) + 3(1-2\nu)\rho^2, \quad \rho = \frac{\operatorname{tr}(\sigma)}{3\bar{\sigma}}, \tag{62}$$

where ρ is the triaxiality ratio. The hardening modulus h is related in this case to the damage D by:

$$g = -\frac{R_\nu \sigma_s^3}{2ZE(1-D)^2}, \tag{63}$$

σ_s being the yield limit which in loading equals $\bar{\sigma}(1-D)$.

For this model $\mathbb{M} = \mathbb{0}$ and we have deviatoric normality. Therefore, Remark 3 applies. The critical conditions of localization, written above in terms of the hardening modulus h, can also be presented here in terms of damage, leading then to a critical value of damage D_c given either by $D_c = 0$ for immediate localization or by D_1, D_3, $D_{1\text{-}3}$ corresponding to h_1, h_3, $h_{1\text{-}3}$, respectively, defined above so that:

$$D_i = 1 - (1+\nu)(1-2\nu)\frac{\sigma_s^3 R_\nu}{ZE^2}\frac{(P_i)^2 + \rho P_i}{2(1-\nu) - 3(1-2\nu)(P_i)^2}, \quad i = 1, 3 \tag{64}$$

$$D_{1\text{-}3} = 1 - \frac{\sigma_s^3 R_\nu}{ZE^2}\frac{1}{18P^2}\left\{2 - 3(1-2\nu)P\rho - 6(1+\nu)P^2 + \sqrt{\left[4 + 18\frac{(1-2\nu)^2}{1-\nu}P^2\rho^2 - 12(1-2\nu)P\rho\right]}\right\}, \tag{65}$$

where $P_3 \leqslant P_2 = P < P_1$ are the eigenvalues of the adimensional deviatoric stress tensor $\mathbb{P} = \mathbb{s}/\bar{\sigma}$.

Depending on the loading history, localization can take place as soon as the initial yield limit is attained, i.e. for $D = 0$, or for an amount of damage $D_c = \min(D_1, D_3, D_{1\text{-}3})$. For the model considered it can be shown that this minimum is always $D_c = D_{1\text{-}3}$.

It is worth noting that the critical value of damage at localization depends not only on P but also on the triaxiality ratio ρ. Figure 9 shows the isovalues of D_c in the plane (ρ, P) for the set of parameters given there. Due to the expression (72) of D_c, this diagram shows a polar symmetry with respect to the origin. The shaded area correspond to $D_c = 0$. In this plane, each proportional stress path is represented by a point and the corresponding critical value of damage is given directly by the isocurve passing by this point. Tension and compression correspond to the points $(\frac{1}{3}, -\frac{1}{3})$ and $(-\frac{1}{3}, \frac{1}{3})$, respectively, while pure shear is represented by the origin. This kind of diagram gives information on the type of loading histories for which localization occurs very late (for example, the area surrounded by the isocurve $D_c = 0.9$) and which can be used as loading programs in experiments designed for identification of constitutive models.

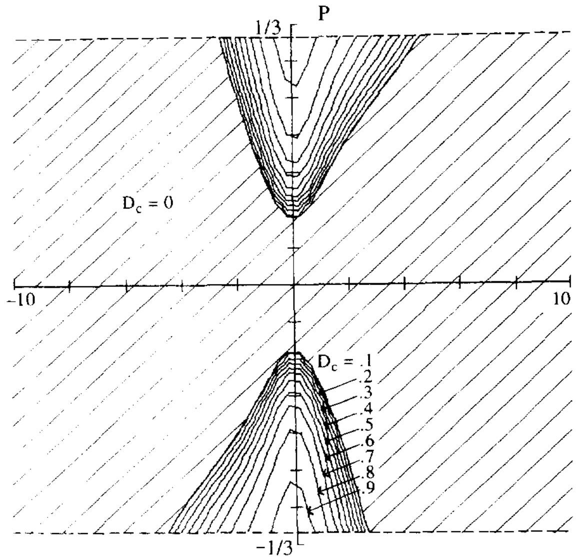

Fig. 9. Critical damage states at localization for elastic–perfectly plastic and damageable material; $E = 200{,}000$ MPa, $\nu = 0.3$, $\sigma_s = 367$ MPa and $Z = 0.03$ MPa.

4.2. *Application to plastic-fracturing models of concrete materials*

The above results are now applied to a general set of constitutive relations used to describe the plastic fracturing behaviour of concrete-like materials. The Hsieh–Ting–Chen model, the Ottosen or the Willam–Warnke models are included. For details regarding these models, see Chapter IV in Chen and Han (1987). The tangent modulus has the general form:

$$\mathbb{H} = \mathbb{E} - \frac{((\alpha/\bar{\sigma})\mathbb{S}+\beta\mathbb{s}+\gamma\bar{\sigma}\mathbb{1})\otimes((\alpha'/\bar{\sigma})\mathbb{S}+\beta'\mathbb{s}+\gamma'\bar{\sigma}\mathbb{1})}{g+[\frac{1}{27}\alpha\alpha'+\frac{2}{3}\beta\beta'+(\beta\alpha'+\alpha\beta')(3P^3-P)](\bar{\sigma}^2/2\mu)+\gamma\gamma'(\bar{\sigma}^2/K)}, \tag{66}$$

where $\mathbb{s}$ is the stress deviator, $\bar{\sigma} = \sqrt{(\frac{3}{2}\mathbb{s}:\mathbb{s})}$ is the equivalent stress, $P = P_2$ is the intermediate eigenvalue of $\mathbb{s}/\bar{\sigma}$ and $\mathbb{S}$ is the gradient of the third invariant $J_3 = \frac{1}{3}(\mathbb{s}\,.\,\mathbb{s}):\mathbb{s}$ with respect to $\boldsymbol{\sigma}$, i.e.

$$\mathbb{S} = \frac{\partial J_3}{\partial \boldsymbol{\sigma}} = \mathbb{s}\,.\,\mathbb{s} - \tfrac{1}{3}\operatorname{tr}(\mathbb{s}\,.\,\mathbb{s})\mathbb{1} \tag{67}$$

and α, β, γ, α', β', γ' are scalar functions of the material parameters and possibly of the stress invariants. This corresponds, in the notations used above, to:

$$\boldsymbol{\alpha} = \frac{\alpha}{\bar{\sigma}}\mathbb{S}+\beta\mathbb{s}, \quad \boldsymbol{\beta} = \frac{\alpha'}{\bar{\sigma}}\mathbb{S}+\beta'\mathbb{s}, \tag{68}$$

$$p = \gamma\bar{\sigma}, \quad q = \gamma'\bar{\sigma}. \tag{69}$$

When localization does not occur at the beginning of the inelastic process, the critical values of g are obtained by substituting eqns (68) and (69) in eqns (26) and (34) and using the relation between h and g. This gives:

$$\frac{g_{ij}}{\mu} = \frac{\bar{\sigma}^2}{4\mu^2(\beta-\alpha P_k)(\beta'-\alpha' P_k)}\left\{-\frac{1+\nu}{2}\left[2\beta\beta' P_k - 2\alpha\alpha'(P_k^2-\tfrac{2}{9}) - \tfrac{2}{9}(\alpha\beta'+\beta\alpha')\right.\right.$$

$$\left. + \frac{1-2\nu}{1+\nu}(\gamma\beta'+\gamma'\beta-(\gamma\alpha'+\gamma'\alpha)P_k)\right]^2$$

$$+\frac{(1-2\nu)^2}{1-\nu^2}\left[\frac{1+\nu}{2(1-2\nu)}(P_k^2-\tfrac{1}{9})(\alpha'\beta-\beta'\alpha)+(\beta\gamma'-\beta'\gamma)+(\alpha'\gamma-\gamma'\alpha)P_k\right]^2$$

$$\left. +\tfrac{9}{4}[(P_k^2-\tfrac{1}{9})(\alpha'\beta-\beta'\alpha)]^2\right\} \tag{70}$$

and

$$\frac{g_k}{\mu} = \frac{\bar{\sigma}^2}{4\mu^2(\beta-\alpha P_k)(\beta'-\alpha' P_k)}\left\{-\frac{1+\nu}{1-\nu}\left[\beta\beta' P_k - \alpha\alpha'(P_k^2-\tfrac{2}{9}) - \tfrac{1}{9}(\alpha\beta'+\beta\alpha')\right.\right.$$

$$\left. - \frac{1-2\nu}{1+\nu}(\gamma\beta'+\gamma'\beta-(\gamma\alpha'+\gamma'\alpha)P_k)\right]^2$$

$$+\frac{(1-2\nu)^2}{1-\nu^2}\left[-\frac{1+\nu}{1-2\nu}(P_k^2-\tfrac{1}{9})(\alpha'\beta-\beta'\alpha)+(\beta\gamma'-\beta'\gamma)+(\alpha'\gamma-\gamma'\alpha)P_k\right]^2$$

$$\left. -[(\beta-\alpha P_k)(\beta'-\alpha' P_k)]^2(\tfrac{4}{3}-3P_k^2)\right\}. \tag{71}$$

In eqns (70) and (71), (i, j, k) is a triplet of distinct numbers belonging to the set (1, 2, 3).

As in the simpler case of Section 4.2, the critical hardening modulus can be expressed as a function of a normalized intermediate component P of the stress deviator. Indeed, due to the deviatoric nature of $\mathbb{P}$, the P_ks can be expressed as a function of P alone:

$$P_1 = -\frac{P}{2}+\sqrt{(\tfrac{1}{3}-\tfrac{3}{4}P^2)};\quad P_2 = P;\quad P_3 = -\frac{P}{2}-\sqrt{(\tfrac{1}{3}-\tfrac{3}{4}P^2)}. \tag{72}$$

For the present model, the non-associativity concerns both the volumetric and deviatoric parts of $\mathbb{\alpha}$ and $\mathbb{\beta}$. For fixed β, $(\beta\gamma'-\beta'\gamma)$ and $(\alpha'\beta-\beta'\alpha)$ can be taken to measure these non-associativities. The non-associativity on the deviatoric part alone may involve localization in the hardening regime ($h > 0$); however, the simultaneous presence of non-associativity on the hydrostatic part can both enhance or delay localization, depending on the loading history considered. A discussion of the role of the two kinds of non-normalities for model (66) can be found in Benallal and Comi (1993).

5. CONCLUSIONS

In the present paper the relation between the structure of the tangent modulus and the critical conditions at localization has been investigated. Using a geometrical method, the problem has been solved completely when the tangent modulus has the form of eqn (2) with $\mathbb{\alpha}$ and $\mathbb{\beta}$ coaxial and for isotropic elastic properties. Explicit formulae are given for the critical hardening modulus and the normal to the critical plane at localization.

When $\mathbb{\alpha}$ (or $\mathbb{\beta}$) has distinct eigenvalues, the normal to the localization plane is always contained in one of their common principal planes and can in some instances be one of their principal directions. This conclusion also holds when these two tensors each have a double eigenvalue corresponding to different principal planes. In the last case, where each of these tensors has a double eigenvalue corresponding to the same principal plane, the normal to the localization plane is either the principal direction corresponding to their single eigenvalue, or arbitrary in the principal plane corresponding to the double eigenvalue, or finally belongs to a cone with axis the principal direction corresponding to the single eigenvalue.

At localization, the critical hardening modulus is shown to have six different expressions in the general case, each of these expressions being valid only in a given region of the constitutive parameters and loading conditions. These regions are also provided explicitly. Simple examples have been presented to show that all these expressions may be of importance in practice.

The results obtained here also show how both deviatoric and hydrostatic non-associativities affect the critical conditions at localization. Both non-associativities may have destabilizing effects in the sense that they anticipate localization.

Finally, the geometrical interpretation of the localization condition given here also holds for the very general case with $\mathbb{a}$ and $\mathbb{b}$ non-coaxial. However, in this case the reduction of the localization analysis to a single plane implies transformations leading to imaginary curves and cannot be used as developed in this paper. Its extension needs more effort and will be provided in a forthcoming paper.

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APPENDIX A

Throughout the paper tensor analysis is used. The symbol $\otimes$ denotes tensor product, the symbol : denotes the doubly contracted product and the symbol . is the scalar product. Outline bold characters (e.g. $\mathbb{N}$ or $\mathbb{\alpha}$) denote tensors; their eigenvalues are marked by indices i, j, k (e.g. N_i, N_j, N_k). Bold characters (e.g. **n**) denote vectors; their components are denoted by indices i, j, k (e.g. n_i, n_j, n_k). Italic characters (e.g. h) denote scalars.

In the presentation of the localization results, the scalars may be affected by one or two indices: one index (e.g. h_i) indicates that the value of the scalar variable h is computed for a situation where the normal to the localization plane is the ith principal direction, while two indices (e.g. $h_{i,j}$) correspond to this normal lying in the plane defined by the ith and the jth principal directions. Indices i, j, k are never summed and the triplet (i, j, k) when used in an expression assumes the values (1, 2, 3), (2, 3, 1) and (3, 1, 2).

APPENDIX B: EXPRESSION OF THE LOCALIZATION CONDITION IN THE (Σ, T) PLANE

The localization condition [eqn (15)] in the (Σ, T) plane reads:

$$\mathcal{F}(T, \Sigma, H) = a\Sigma^2 + bT^2 + c\Sigma T + d\Sigma + eT + f(H) \geqslant 0, \tag{A1}$$

where

$$a = (\lambda+\mu)\cdot\left[\left(\frac{M_1N_1}{(N_1-N_2)(N_1-N_3)}+\frac{M_2N_2}{(N_2-N_1)(N_2-N_3)}+\frac{M_3N_3}{(N_3-N_1)(N_3-N_2)}\right)^2-1\right] \tag{A2}$$

$$b = (\lambda+\mu)\cdot\left[\left(\frac{M_1}{(N_1-N_2)(N_1-N_3)}+\frac{M_2}{(N_2-N_1)(N_2-N_3)}+\frac{M_3}{(N_3-N_1)(N_3-N_2)}\right)^2\right] \tag{A3}$$

$$c = 2\sqrt{[b(a+\lambda+\mu)]} \tag{A4}$$

$$\begin{aligned} d = 2\mu\rho-(\lambda+2\mu)\cdot&\left(\frac{M_1^2N_1}{(N_1-N_2)(N_1-N_3)}+\frac{M_2^2N_2}{(N_2-N_1)(N_2-N_3)}+\frac{M_3^2N_3}{(N_3-N_1)(N_3-N_2)}\right)\\ &+2\left(\frac{M_1N_1}{(N_1-N_2)(N_1-N_3)}+\frac{M_2N_2}{(N_2-N_1)(N_2-N_3)}+\frac{M_3N_3}{(N_3-N_1)(N_3-N_2)}\right)\\ &\times\left[(\lambda+\mu)\left(\frac{M_1N_2N_3}{(N_1-N_2)(N_1-N_3)}+\frac{M_2N_1N_3}{(N_2-N_1)(N_2-N_3)}+\frac{M_3N_1N_2}{(N_3-N_1)(N_3-N_2)}\right)-\mu\rho\right] \end{aligned} \tag{A5}$$

$$\begin{aligned} e = (\lambda+2\mu)\cdot&\left(1-\frac{M_1^2}{(N_1-N_2)(N_1-N_3)}+\frac{M_2^2}{(N_2-N_1)(N_2-N_3)}+\frac{M_3^2}{(N_3-N_1)(N_3-N_2)}\right)\\ &+2\left(\frac{M_1}{(N_1-N_2)(N_1-N_3)}+\frac{M_2}{(N_2-N_1)(N_2-N_3)}+\frac{M_3}{(N_3-N_1)(N_3-N_2)}\right)\\ &\times\left[(\lambda+\mu)\left(\frac{M_1N_2N_3}{(N_1-N_2)(N_1-N_3)}+\frac{M_2N_1N_3}{(N_2-N_1)(N_2-N_3)}+\frac{M_3N_1N_2}{(N_3-N_1)(N_3-N_2)}\right)-\mu\bar{\rho}\right] \end{aligned} \tag{A6}$$

$$\begin{aligned} f = \mu(\rho^2-\bar{\rho}^2)-4\mu(\lambda+2\mu)H-(\lambda+2\mu)&\left(\frac{M_1^2N_2N_3}{(N_1-N_2)(N_1-N_3)}+\frac{M_2^2N_1N_3}{(N_2-N_1)(N_2-N_3)}+\frac{M_3^2N_1N_2}{(N_3-N_1)(N_3-N_2)}\right)\\ +(\lambda+\mu)&\left(\frac{M_1N_2N_3}{(N_1-N_2)(N_1-N_3)}+\frac{M_2N_1N_3}{(N_2-N_1)(N_2-N_3)}+\frac{M_3N_1N_2}{(N_3-N_1)(N_3-N_2)}\right)^2\\ &-2\mu\rho\left(\frac{M_1N_2N_3}{(N_1-N_2)(N_1-N_3)}+\frac{M_2N_1N_3}{(N_2-N_1)(N_2-N_3)}+\frac{M_3N_1N_2}{(N_3-N_1)(N_3-N_2)}\right). \end{aligned} \tag{A7}$$

$\mathcal{F} = 0$ then represents a conical curve. The discriminant of this expression is:

$$\Delta = -\left[2(\lambda+\mu)\left(\frac{M_1}{(N_1-N_2)(N_1-N_3)}+\frac{M_2}{(N_2-N_1)(N_2-N_3)}+\frac{M_3}{(N_3-N_1)(N_3-N_2)}\right)\right]^2. \tag{A8}$$

As Δ is always non-positive, this curve is either a hyperbola when $N_k\,(M_i-M_j) \neq M_k\,(N_i-N_j)$ or a parabola with vertical axis when $N_k\,(M_i-M_j) = M_k\,(N_i-N_j)$. This last condition is fulfilled in particular if $M_i = N_i$ $(i = 1, 2, 3)$ or $M_i = -N_i$ $(i = 1, 2, 3)$, in the case of multiple eigenvalues ($M_i = M_j$ and $N_i = N_j$) or for deviatoric associativity ($\mathbb{M} = \mathbb{O}$).